

An electric-resistance approach to return time

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Abstract

A new proof is given for the formula for the expected return time of a random walk on a graph. This proof makes use of known relationships between electric resistance and random walks.

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Let G be a finite, connected graph with n vertices and m edges. We will use the standard notation \sim to denote adjacency in the graph, and for $z \in G$ let $\deg(z)$ denote the degree of z . Let X_j denote simple random walk on G ; that is, X_j is the Markov chain taking values in the vertex set of G with transition probabilities given by

$$(1) \quad P(X_{j+1} = z | X_j = y) = \begin{cases} \frac{1}{\deg(y)} & \text{if } y \sim z \\ 0 & \text{otherwise} \end{cases}.$$

Let $T_z = \inf\{j \geq 0 : X_j = z\}$, and let $T_z^+ = \inf\{j \geq 1 : X_j = z\}$. The object of interest for us is the expected return time, $E_z[T_z^+]$. The following elegant theorem is well known.

Theorem 1

$$(2) \quad E_z[T_z^+] = \frac{2m}{\deg(z)}.$$

This result admits a considerable generalization. Consider each edge (y, z) as a wire in a circuit with a given conductance C_{yz} , which is a nonnegative number which measures how easily electricity (and the random walk) passes along the edge. For each vertex z let $C_z = \sum_{y \sim z} C_{yz}$. Let X_j now be the Markov chain taking values in the vertex set of G with transition probabilities given by

$$(3) \quad P(X_{j+1} = z | X_j = y) = \begin{cases} \frac{C_{yz}}{C_y} & \text{if } y \sim z \\ 0 & \text{otherwise} \end{cases}.$$

This is the random walk induced by the electric network, and simple random walk corresponds to taking conductances of 1 (or any positive constant) across each edge. It should be noted that this construction is in fact quite general, since any reversible Markov chain can be realized as such an induced random walk (see [LPW09, Ch. 9]). Let $C = \sum_{y \in G} C_y$. We then have the following extension of Theorem 1 (which in fact applies to infinite graphs as well under the assumption that C is finite).

Theorem 2

$$(4) \quad E_z[T_z^+] = \frac{C}{C_z}.$$

The standard method of proving Theorem 2 is to appeal to a result from Markov chain theory, namely that an irreducible Markov chain with a stationary distribution π satisfies $E_z[T_z^+] = 1/\pi_z$; and then simply verifying that $\pi_z = \frac{C_z}{C}$ is the stationary distribution for the chain X_j (see [Nor98, Sec. 1.7]). On the other hand, researchers studying electric resistance have uncovered many identities and bounds on such quantities as hitting times, commute times, and cover times ([CRR⁺96]); mixing times ([AF02, Ch. 4]); and edge-cover times ([GW12]). It is therefore natural to search for a derivation of Theorems 1 and 2 which makes more use of the principles which relate electric resistance to random walks, especially in light of the statement of the second theorem. We now present such a proof, naturally of the more general result, Theorem 2.

Fix $z \in G$, and construct a new graph \tilde{G} which contains G as a subgraph by adding a vertex \tilde{z} to G with a single edge connecting \tilde{z} to z . Across the new

edge lay a conductance of 1. Let \tilde{X}_m denote a random walk on \tilde{G} induced by the conductances present (the original ones in G , together with the edge with unit conductance connecting z and \tilde{z}). For $x, y \in \tilde{G}$, let $\tilde{R}_{x,y}$ be the effective resistance within \tilde{G} between x and y , and let $\tilde{T}_y = \inf\{j \geq 0 : \tilde{X}_j = y\}$. Let \tilde{C} be twice the sum of the conductances across all edges in \tilde{G} ; note that $\tilde{C} = C + 2$. It is known ([AF02, Cor. 11, Ch. 3]) that

$$(5) \quad E_{\tilde{z}}[\tilde{T}_z] + E_z[\tilde{T}_{\tilde{z}}] = \tilde{C}\tilde{R}_{z,\tilde{z}}.$$

However, it is trivial that $E_{\tilde{z}}[\tilde{T}_z] = 1$ (the walk beginning at \tilde{z} has no choice but to pass to z at time 1), and it is equally trivial that $R_{z,\tilde{z}} = 1$ (there are no paths from \tilde{z} to z except for along the edge connecting them). Making the necessary substitutions yields

$$(6) \quad E_z[\tilde{T}_{\tilde{z}}] = C + 1.$$

Now, at time $\tilde{T}_{\tilde{z}} - 1$ the walk \tilde{X}_m must necessarily reside at z . Furthermore, at each visit to z the walk \tilde{X}_m has probability $\frac{1}{C_z+1}$ of passing to \tilde{z} at the next step, and between visits to z the walk performs excursions within G , which will each take an average of $E_z[T_z^+]$ steps. The number of excursions within G before $\tilde{T}_{\tilde{z}}$ is a Bernoulli trial with probability $\frac{1}{C_z+1}$, and as is well known the expected number of such trials until first success is $C_z + 1$; however the number of excursions will in fact be one less than the number of visits to z , since the walk begins at z . It follows then that the expected number of excursions within G before $\tilde{T}_{\tilde{z}}$ will be C_z . Adding 1 to record the final step from z to \tilde{z} we obtain

$$(7) \quad E_z[\tilde{T}_{\tilde{z}}] = C_z E_z[T_z^+] + 1.$$

Equating the right-hand sides of (7) and (6) yields (4).

Remark: The key idea of attaching a new vertex to a vertex z in a graph and starting a random walk there, armed with the knowledge that the first step must be to z , appears also in different contexts in [NP95, Lemma 3.1] and [LPW09, Ex. 10.4].

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